

The vector space of linear maps

1. Suppose you have two $A, B: \mathbf{R}^4 \rightarrow \mathbf{R}^3$ defined as follows:

$$A = \begin{pmatrix} -0.38 & 0.64 & 0.90 & 0.21 \\ 0.26 & 0.37 & 0.71 & 0.66 \\ -0.49 & 0.94 & 0.18 & -0.50 \end{pmatrix} \text{ and } B = \begin{pmatrix} -0.05 & 0.59 & 0.02 & 0.08 \\ -0.25 & 0.99 & -0.26 & 0.68 \\ -0.23 & 0.48 & -0.09 & 0.27 \end{pmatrix}.$$

Find $3A + 2B$.

2. Given the vector $\mathbf{u} = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 3 \end{pmatrix}$, suppose from Question 1, you determine $A\mathbf{u} = \begin{pmatrix} -0.51 \\ 2.47 \\ -4.18 \end{pmatrix}$ and $B\mathbf{u} = \begin{pmatrix} -1.02 \\ -0.70 \\ -0.70 \end{pmatrix}$.

Find both $(3A + 2B)\mathbf{u}$ and $3A\mathbf{u} + 2B\mathbf{u}$, the first by multiplying the matrix you found in Question 1 and the vector $\mathbf{u}$, and the second by finding the linear combination of the two vectors $A\mathbf{u}$ and $B\mathbf{u}$. Should they be the same?

3. What is the dimension of all linear maps $A: \mathbf{R}^4 \rightarrow \mathbf{R}^3$? That is, what is the dimension of $\mathcal{L}(\mathbf{R}^4, \mathbf{R}^3)$?

4. What would be a *canonical basis* for $\mathcal{L}(\mathbf{R}^4, \mathbf{R}^3)$? Recall that each element in this vector space is a 3×4 matrix, and each matrix should be able to be written as a simple linear combination of basis vectors.

5. We previously discussed how an image can be thought of as a vector, where values less than or equal to zero are interpreted as black, and values greater than or equal to 1 are interpreted as white, and everything else is a relative shade of gray. Multiplying such an image by a scalar greater than 1 would be brightening the image, while multiplying such an image by a scalar between 0 and 1 would be darkening the image. For example, a shrunk-down grayscale image of the flag of Benin may look as follows:

$$\begin{pmatrix} 0.3 & 0.3 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.3 & 0.3 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.3 & 0.3 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.3 & 0.3 & 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}$$

Does it make any sense to think of such an image as a map $A: \mathbf{R}^6 \rightarrow \mathbf{R}^4$?

6. Is it true that if $A: \mathbf{R}^5 \rightarrow \mathbf{R}^7$ that $A + (-1)A = 0A$?

7. Given a function, the identity operator is $\hat{1}f = f$ so that $(\hat{1}f)(x) = f(x)$. Two other operators are the derivative and second-derivative operators, $\frac{d}{dx}f(x)$ and $\frac{d^2}{d^2x}f(x)$, respectively. Is $\frac{d^2}{d^2x} + 3\frac{d}{dx} + 2\hat{1}$ a linear operator, and if so, what is $\left(\frac{d^2}{d^2x} + 3\frac{d}{dx} + 2\hat{1}\right)\sin(x)$?